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Limit cycles for a class of second order differential equations.

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Abstract

We study the limit cycles of a wide class of second order differential equations, which can be seen as a particular perturbation of the harmonic oscillator. In particular, by choosing adequately the perturbed function we show, using the averaging theory, that it is possible to obtain as many limit cycles as we want.

Keywords: Limit cycle, second order differential equation, averaging theory.

1. Introduction and statement of the main results

The second order differential equations arise in many areas of science and technology essentially, but in the last years many important applications in social science, economics and business administration have raised, as well as plenty of applications in physics, chemistry, biology and the engineries. Nevertheless, in general is impossible to determine the solution of a second order differential equation in terms of explicit functions, so we look for properties of these kind of equations which can be obtained without solving it. The problem is more relevant in perturbation theory, where the periodic orbits play a main role. For more information about the study of the periodic orbits of second order differential equations see for instance the references [6, 7, 8, 14, 15], and the ones quoted there.

It is well known that all solutions of the harmonic oscillator are periodic with the same period, i.e. the origin of this system is an isochronous center. When we take a perturbation of the harmonic oscillator, which periodic orbits persist? which periodic orbits generate limit cycles? In this paper we ask these questions for a wide class of second order differential equations, which are essentially a perturbation of the harmonic oscillator. More precisely, we study equations of the form

$$\ddot{x} = -(1 + \varepsilon\kappa\cos^2 t)x + \varepsilon f(t, x, \dot{x}), \tag{1}$$

where ε is a small parameter, κ is a real parameter and the function $f(t, x, \dot{x})$ is at least of class C^2 and 2π–periodic in the variable t in order to apply the averaging theory [9].

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We would like to mention the Ermakov systems, which have a long history in sciences and many important applications in Physics. In the last years many people have retook their study, see for instance [2, 4, 5]. Since the classical Ermakov system has the form

$$\ddot{x} = -\omega^2(t)x + \frac{1}{x^3}G(x).$$

The class of second order differential equations (1) contains the subclas of Ermakov systems given by the functions $G(x) = \varepsilon x^3 f(x)$ and $\omega^2(t) = 1 + \varepsilon\kappa \cos^2 t$.

The first goal of this paper is provide a general result which, for ε sufficiently small, allows to study the periodic orbits of the second differential equation (1) for an arbitrary function $f(t, x, \dot{x})$, see the next Theorem 1. After, we shall compute explicitly the periodic orbits of such second order differential equations for some explicit functions $f(t, x, \dot{x})$, see the next Propositions, 2 and 3.

For stating our results we need some definitions. Thus we define the functions

$$\begin{aligned} h_1(s,r,a) = \ & \kappa r \cos s \cos^2(a+s)\sin(a+s) - \\ & f(a+s, r\cos s, -r\sin s)\sin(a+s), \\ h_2(s,r,a) = \ & -\kappa r \cos s \cos^3(a+s) + \\ & f(a+s, r\cos s, -r\sin s)\cos(a+s), \end{aligned}$$

and

$$\mathsf{f}_j(r,a) = \frac{1}{2\pi}\int_0^{2\pi} h_j(s,r,a)ds \quad \text{for} \quad j = 1, 2.$$

We say that (r_0, a_0) is a *simple solution* of system

$$\mathsf{f}_1(r,a) = 0, \qquad \mathsf{f}_2(r,a) = 0, \tag{2}$$

if the determinant $\left(\dfrac{\partial(\mathsf{f}_1,\mathsf{f}_2)}{\partial(r,a)}\right)(r_0, a_0) \neq 0$.

We recall that a *limit cycle* of the second order differential equation (1) is a periodic orbit isolated in the set of all periodic orbits of (1).

Now we can state the main result of this paper.

Theorem 1. *Let (r_0, a_0) be a simple solution of system (2). Then for $\varepsilon \neq 0$ sufficiently small the second order differential equation (1) has a limit cycle $x_\varepsilon(t)$, such that when $\varepsilon \to 0$ it tends to the 2π–periodic solution $r_0\cos(a_0 - t)$ of system (1) with $\varepsilon = 0$.*

Theorem 1 will be proved in section 3.

This theorem allow to determine the existence of limit cycles of the second order differential equation (1) with $\varepsilon \neq 0$ sufficiently small, and the periodic orbits of the harmonic oscillator (1) with $\varepsilon = 0$ from which they bifurcate. To show the powerful of Theorem 1 we give some applications of it, we obtain that by choosing conveniently the perturbed function, equation (1) can have one, or as many limit cycles as we want. The following propositions, whose proofs will be given in section 4, show these different possibilities.

Proposition 2. *For $\varepsilon \neq 0$ sufficiently small and for all $\kappa > 0$ the second order differential equation* (1) *with $f(t, x, \dot{x}) = \cos t$ has a unique limit cycle $x_\varepsilon(t)$ such that $x_\varepsilon(t) \to \dfrac{4}{3\kappa}\cos t$ as $\varepsilon \to 0$.*

Proposition 3. *For $\varepsilon \neq 0$ sufficiently small and $\kappa \in [0, 1]$, the second order differential equation* (1) *with $f(t, x, \dot{x}) = \sin x$ can have as many limit cycles as we want.*

2. A basic result from averaging theory

In order to have a self contained paper, in this section we present the basic results from the averaging theory that are necessary for proving the main result of this paper, i.e Theorem 1.

We considerer a differential system of the form

$$\dot{\mathbf{x}}(t) = F_0(t, \mathbf{x}) + \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \tag{3}$$

where the functions $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are of class C^2 in Ω and T–periodic in the variable t, Ω is an open subset of $\mathbb{R}^n$. When $\varepsilon = 0$ we get the unperturbed system

$$\dot{\mathbf{x}}(t) = F_0(t, \mathbf{x}), \tag{4}$$

we assume that the above unperturbed system has an open subset of Ω fulfilled of periodic solutions. For ε sufficiently small is possible obtain periodic solutions of system (3) by using the averaging theory.

Let $\mathbf{x}(t, \mathbf{z}, \varepsilon)$ be the solution of system (3) such that $\mathbf{x}(0, \mathbf{z}, \varepsilon) = \mathbf{z}$. The variational equation of the unperturbed system (4) along the periodic solution $\mathbf{x}(t, \mathbf{z}, 0)$ is

$$y' = D_{\mathbf{x}} F_0(t, \mathbf{x}(t, \mathbf{z}, 0))y. \tag{5}$$

where y is a $n \times n$ matrix. Let $M_{\mathbf{z}}(t)$ be a fundamental matrix of the linear differential system (5). We assume that Ω contains an open subset W formed only by periodic orbits, all of them having the same period T, i.e. W is an *isochronous set* of system (3). Then we have the following result.

Theorem 4. *Assume that Ω contains an open and bounded set W such that $Cl(W) \subset \Omega$ and for each $\boldsymbol{z} \in Cl(W)$, the solution $x(t, \boldsymbol{z}, 0)$ is T–periodic. Let $\mathsf{F} : W \to \mathbb{R}^2$ be the function defined by*

$$\mathsf{F}(\boldsymbol{z}) = \frac{1}{T}\int_0^T M_{\boldsymbol{z}}^{-1}(t) F_1(t, \boldsymbol{x}(t, \boldsymbol{z}, 0))dt. \tag{6}$$

If there exist $\boldsymbol{p} \in W$ with $\mathsf{F}(\boldsymbol{p}) = 0$ and the determinant of $(d\mathsf{F}/d\boldsymbol{z})(\boldsymbol{p}) \neq 0$, then there exists a T–periodic solution $\phi(t, \varepsilon)$ of system (3) *such that $\phi(0, \varepsilon) \to \boldsymbol{p}$ as $\varepsilon \to 0$.*

For a shorter proof of Theorem 4 see Corollary 1 of [3]. In fact this result goes back to Malkin [12] and Roseau [13]. For an application of Theorem 4 to Hamiltonian systems in a more general context you can see [10].

3. Proof of Theorem 1

For proving Theorem 1 we shall use Theorem 4. In what follows we will use the notation introduced in section 2.

Proof of Theorem 1. The second order differential equation (1) can be written as the first order differential system

$$\begin{aligned} \dot{x} &= v, \\ \dot{v} &= -x + \varepsilon(-\kappa x \cos^2 t + f(t,x,v)) = -x + \varepsilon f_1(t,x,v). \end{aligned} \tag{7}$$

This system can be written in the vectorial form

$$\dot{\mathbf{x}} = F_0(t,x,v) + \varepsilon F_1(t,x,v), \tag{8}$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ v \end{pmatrix}, \quad F_0(t,x,v) = \begin{pmatrix} v \\ -x \end{pmatrix} \quad \text{and} \quad F_1(t,x,v) = \begin{pmatrix} 0 \\ f_1(t,x,v) \end{pmatrix}.$$

The unperturbed part of system (7) is

$$\begin{aligned} \dot{x} &= v, \\ \dot{v} &= -x, \end{aligned} \tag{9}$$

it corresponds to the simple harmonic oscillator whose solution is

$$\begin{aligned} x(t) &= x_0 \cos t + v_0 \sin t, \\ v(t) &= -x_0 \sin t + v_0 \cos t. \end{aligned}$$

We observe that for any initial condition $(x_0, v_0) \in \mathbb{R}^2 \setminus \{(0,0)\}$, the corresponding solution is 2π–periodic and can be write as

$$x(t) = r\cos(a-t), \qquad v(t) = r\sin(a-t), \tag{10}$$

where $r > 0$ and a are constants which can be written in terms of the initial conditions (x_0, v_0). We say that the origin of system (9) is an *isochronous center*.

Let $\mathbf{x}(t,\mathbf{z},\varepsilon)$ be the solution of system (7) such that $\mathbf{x}(0,\mathbf{z},\varepsilon) = \mathbf{z} = (r,a)$. The variational equation of the unperturbed system (9) along the periodic solution $\mathbf{x}(t,\mathbf{z},0)$ is

$$y' = D_{\mathbf{x}} F_0(t, \mathbf{x}(t,\mathbf{z},0))y, \tag{11}$$

where y is a 2×2 matrix. Let $M(t)$ be a fundamental matrix of the linear differential system (11), it takes the simple form

$$M(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Now, evaluating the function $f_1(t,x,v)$ along the solution (10) of the unperturbed system we obtain the function

$$h(t,r,a) = -\kappa r \cos(a-t)\cos^2 t + f(t, r\cos(a-t), r\sin(a-t)).$$

Then we have

$$M^{-1}(t)F_1(t,\mathbf{x}(t,\mathbf{z}))) = \begin{pmatrix} -h(t,r,a)\sin t \\ h(t,r,a)\cos t \end{pmatrix}.$$

Doing the change of variable $t \to a+s$ the above expression becomes

$$\begin{pmatrix} \kappa r\cos s\cos^2(a+s)\sin(a+s) - f(a+s, r\cos s, -r\sin s)\sin(a+s) \\ -\kappa r\cos s\cos^3(a+s) + f(a+s, r\cos s, -r\sin s)\cos(a+s) \end{pmatrix}.$$

We observe that the first part in the two previous expressions does not depend of the function $f(a+s, r\cos s, -r\sin s)$.

Let

$$\mathsf{F}(\mathbf{z}) = \begin{pmatrix} \mathsf{f}_1(r,a) \\ \mathsf{f}_2(r,a) \end{pmatrix} = \frac{1}{2\pi}\int_0^{2\pi} M^{-1}(t)F_1(t,\mathbf{x}(t,\mathbf{z})))dt. \tag{12}$$

Let $W \neq \emptyset$ be an open and bounded subset of $\mathbb{R}^2$, formed only by periodic orbits, all of them having the same period 2π, that is W is a isochronous set for system (9).

Let $(r_0, a_0) \in W$ a simple solution of the system

$$\mathsf{f}_1(r,a) = 0, \qquad \mathsf{f}_2(r,a) = 0, \tag{13}$$

that is determinant $\left(\dfrac{\partial(\mathsf{f}_1,\mathsf{f}_2)}{\partial(r,a)}\right)(r_0,a_0) \neq 0$.

With all the above we have verified the hypotheses of Theorem 4, then the proof of Theorem 1 follows from it. $\square$

4. Proof of the propositions

In this section we present the proof of the propositions stated in section 1, showing that depending on the function $f(t,x,\dot{x})$ in (1), it is possible to obtain as many limit cycles as we want. Again we will use the notations introduced in section 2 and in the proof of Theorem 1 of section 3.

Proof of Proposition 2. Taking $f(t,x,v) = \cos t$, the second order differential equation (1) becomes

$$\ddot{x} = -(1+\kappa\varepsilon\cos^2 t)x + \varepsilon\cos t. \tag{14}$$

By straightforward computations the value of the integral (12) for the above $f(t,x,v)$ at the initial condition (r,a) is

$$\mathsf{F}(r,a) = \begin{pmatrix} \kappa r\sin a/8 \\ (4-3\kappa r\cos a)/8 \end{pmatrix},$$

which has a unique zero for $r > 0$ given by $(r,a) = (4/(3\kappa), 0)$ with $\kappa > 0$. The determinant of $(d\mathsf{F}/d\mathbf{z})\,(4/(3\kappa),0) = \dfrac{1}{16\kappa} \neq 0$. Then applying Theorem 1, for $\kappa > 0$ we can assure the existence of a unique 2π–periodic solution $\mathbf{x}_\varepsilon(t)$ of system (14) such that

$$x_\varepsilon(t) \to \frac{4}{3\kappa}\cos t \quad \text{as} \quad \varepsilon \to 0.$$

So Proposition 2 is proved. $\square$

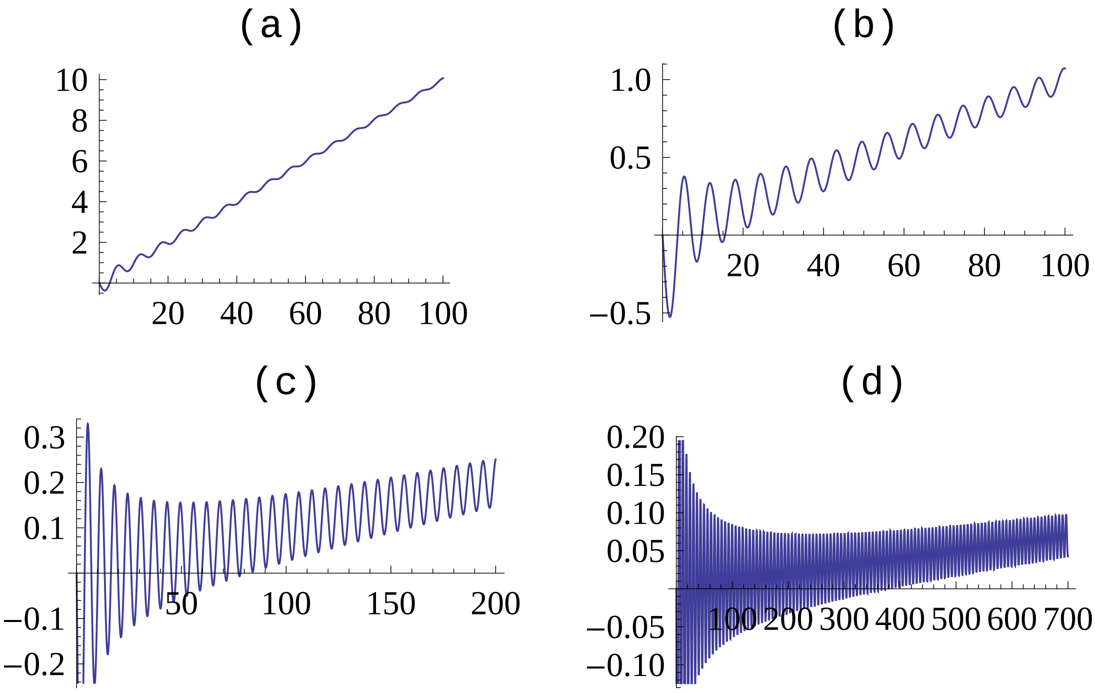

Figure 1: Graphs of the functions $\kappa r - J_1(r)\sin a^*$ for (a) $\kappa = 0.1$; (b) $\kappa = 0.01$; (c) $\kappa = 0.001$ and (d) $\kappa = 0.0001$.

Proof of Proposition 3. In this case the function $f(t, x, v) = \sin x$ depends only on x. So we must analyze the differential system

$$
\begin{aligned}
\dot{x} &= v, \\
\dot{v} &= -(1 + \varepsilon\kappa\cos^2 t)x + \varepsilon \sin x.
\end{aligned}
$$

Evaluating the function $f_1(t, x, v) = -x\kappa\cos^2 t + \sin x$ along the 2π–periodic solutions (10) of the unperturbed system, we obtain

$$
h(t, r, a) = -\kappa r \cos(a - t)\cos^2 t + \sin(r\cos(a - t)),
$$

therefore $M^{-1}(t)F_1(t, \mathbf{x}(t, (r, a)))$ becomes

$$
\begin{pmatrix} -\sin t\left(-\kappa r\cos(a - t)\cos^2 t + \sin(r\cos(a - t))\right) \\ \cos t\left(-\kappa r\cos(a - t)\cos^2 t + \sin(r\cos(a - t))\right) \end{pmatrix}.
$$

Now doing the change of variable $t \to a + s$ we have

$$
\begin{pmatrix} \kappa r\cos s\cos^2(a + s)\sin(a + s) - \sin(a + s)\sin(r\cos s) \\ -\kappa r\cos s\cos^3(a + s) + \cos(a + s)\sin(r\cos s), \end{pmatrix}
$$

Finally integrating these expressions with respect to the variable s from 0 to 2π, dividing by 2π and assuming that $r > 0$, we obtain

$$
\begin{pmatrix} \mathsf{f}_1(r, a) \\ \mathsf{f}_2(r, a) \end{pmatrix} = \frac{1}{8}\begin{pmatrix} \kappa r - J_1(r)\sin a \\ -3\kappa r + 8J_1(r)\cos a, \end{pmatrix}
$$

where $J_1(r)$ is the Bessel function of the first kind, for a precise definition see [1].

The system $\mathsf{f}_1(r,a) = \mathsf{f}_2(r,a) = 0$ is equivalent to the system

$$8\cos a = 3\sin a, \quad \kappa r - J_1(r)\sin a = 0.$$

Taking $a = a^*$ one of the values of $\arctan(8/3)$, the previous system reduces to the unique equation

$$\kappa r - J_1(r)\sin a^* = 0. \tag{15}$$

Then the simple solutions of the system $\mathsf{f}_1(r,a) = \mathsf{f}_2(r,a) = 0$ correspond to the simple solutions of equation (15), which depend on the value of the parameter κ. For instance, for $\kappa = 1$ the above system has exactly one simple zero at $r = 0$, for $\kappa = 0$ it corresponds to the Bessel function which has infinitely many simple zeroes. It is not difficult to verify that for $\kappa \in (0,1)$, we can have as many simple solutions as we want. In Figure 1 we have plot the graphs of some functions $\kappa r - J_1(r)\sin a^*$ changing the value of κ. The proof of Proposition 3 is complete. $\square$

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